Analytical Studies on a Modified Nagel-Schreckenberg Model with the Fukui-Ishibashi Acceleration Rule

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We propose and study a one-dimensional traffic flow cellular automaton model of high-speed vehicles with the Fukui-Ishibashi-type (FI) acceleration rule for all cars, and the Nagel-Schreckenberg-type (NS) stochastic delay mechanism. By using the car-oriented mean field theory, we obtain analytically the fundamental diagrams of the average speed and vehicle flux depending on the vehicle density and stochastic delay probability. Our theoretical results, which may contribute to the exact analytical theory of the NS model, are in excellent agreement with numerical simulations.

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I. INTRODUCTION

Traffic flow that displays various complex behaviors is a kind of many-body systems of strongly interacting vehicles. One of the approaches to microscopic traffic processes is based on cellular automaton (CA) [1], which treats the motions of cars as hopping processes on one-dimensional lattices. In the past few decades, CA models for traffic flow have attracted much interest of a community of physicists [1, 2]. Compared with other dynamical approaches, for instance the fluid dynamical approach, CA models are conceptually simpler, and can be easily implemented on computers for numerical investigations [3].

Two popular one-dimensional (1D) CA models are the Nagel-Schreckenberg (NS) model [4] and the Fukui-Ishibashi (FI) model [5], where periodic boundary conditions are used to mimic the traffic flow on highway. An exact car-oriented mean field theory (COMF) have been developed to study the FI model, with an arbitrary limit on the maximum speed v_{max}, car density ρ, and the delay probability f [6]. So far as we know, however, there has been no established exact analytical theory for the NS model, due to the complications in the time evolution of the flow caused by the acceleration and stochastic delay rules. In order to comprehend how these rules affect the evolution and the corresponding asymptotic steady state, we study a 1D traffic flow CA model which combines the NS stochastic delay rule and the FI acceleration mechanism.

This paper is organized as follows. In Sec. II, we give the definition of the model and the evolution equations for the inter-car spacings. Next, we present the fundamental diagrams of the average speed and vehicle flux depending on the vehicle density and stochastic delay probability, compared with the numerical simulations. In Sec. IV, we summary with a discussion of our results in connection with the FI and NS models.

II. THE MODEL AND ANALYTICAL SOLUTION OF ASYMPTOTIC AVERAGE VELOCITY

The modified Nagel-Schreckenberg model with the Fukui-Ishibashi acceleration rule is a probabilistic automaton, in which space and time are discrete and hence also the velocities. The road of length L is divided into cells of certain length, each of which can either be empty or occupied by just one car. The state of the nth car ($n = 1, \ldots, N$) is characterized by the momentary velocity $v_n(t)$; in this model, the maximum speed is fixed as $v_{max} = M$, and the vehicle density is $\rho = N/L$. Let $C_n(t)$ represent the number of empty sites in front of the nth vehicle at the time step t, then we have

$$C_n(t+1) = C_n(t) + v_{n+1}(t) - v_n(t). \tag{1}$$

As a function of the inter-car spacing $C_n(t)$ and the stochastic delay probability f, the velocity of the nth car at time step t can be written as:

$$v_n(t) = F_M(f, C_n(t)), \tag{2}$$

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where

$$F_M(f,C) = \begin{cases} 0, & \text{if } C = 0 \\ C-1 & \text{with probability } f, \text{ if } 0 < C < M \\ C & \text{with probability } 1-f, \text{ if } 0 < C < M \\ M-1 & \text{with probability } f, \text{ if } C \geq M \\ M & \text{with probability } 1-f, \text{ if } C \geq M. \end{cases} \quad (3)$$

Here, as one can see, we have approximately adopted the acceleration rule introduced by Fukui and Ishibashi. The FI model can be considered as a NS model for "aggressive driving", since the rules of the FI are nearly identical to the NS model, except that the acceleration rule has been changed from "the vehicle speed is at most increased by 1 at each step" to "every car accelerates to v_{max}" (which is adopted in the present model) and the stochastic delay mechanism has been modified as "only cars with v_{max} will delay stochastically" (which is not adopted here). For $v_{max}=1$, this may not change anything, however, for higher velocities it will lead to a considerable enhancement of the flow. Although the model is less realistic than the NS, it is still of interest due to its simplicity. The analytical description of the present model is of course much simpler than that of the NS, and hence it might serve as a testing ground for new analytical methods as well as new models.

If $N_i(t)$ represents the number of inter-car spacings with length i at time t, then the probability of finding such a spacing at time t is $P_i(t) = N_i(t)/N$. Hereafter, $P_i(t)$ will indicated by P_i for simplicity, except if specified otherwise. Let Q_i denote the probability that an arbitrary vehicle moves i sites during a given time step, then we have:

$$\begin{aligned} Q_0 &= P_0 + fP_1 \\ Q_i &= P_i(1-f) + fP_{i+1}(1 \leq i \leq M-2) \\ Q_{M-1} &= P_{M-1}(1-f) + f\sum_{k=M}^{L-N} P_k \\ Q_M &= (1-f)\sum_{k=M}^{L-N} P_k. \end{aligned} \quad (4)$$

Obviously, $P_i = 0$ when $i > L-N$, since the maximum inter-car spacing must be less than $L-N$. To obtain the nonvanishing P_i, we introduce $W_{i \to j}$ to denote the probability of finding an inter-car spacing with length i at a given time which changes into length j at the next time. According to the Eqs. (1)-(4), one can write all the nonzero $W_{i \to j}$ as follows:

$$\begin{aligned} W_{0 \to j} &= P_0 Q_j, \quad (1 \leq j \leq M) \\ W_{i \to 0} &= P_i(1-f)Q_0, \quad (1 \leq i \leq M-1) \\ W_{i \to j} &= P_i[fQ_{j-1} + (1-f)Q_j], \quad (1 \leq i \leq M-1, 1 \leq j \leq M, i \neq j) \\ W_{i \to M+1} &= P_i f Q_M, \quad (1 \leq i \leq M-1) \\ W_{i \to i-M} &= P_i(1-f)Q_0, \quad (i \geq M) \\ W_{i \to i-(M-j)} &= P_i[fQ_{j-1} + (1-f)Q_j], \quad (i \geq M, 1 \leq j \leq M-1) \\ W_{i \to i+1} &= P_i f Q_M, \quad (i \geq M) \\ W_{i \to j} &= 0, \quad \text{otherwise.} \end{aligned} \quad (5)$$

When the system approaches its asymptotic steady state, all the P_j will cease to change; thus the detailed statistical equilibrium condition for the steady state holds:

$$\sum_{i \neq m} W_{i \to m} = \sum_{i \neq m} W_{m \to i}, \quad \forall m. \quad (6)$$

Since there are $L-N+1$ variables $P_0, P_1, \cdots P_{L-N}$ in Eqs. (6), which satisfy the following two normalization equations:

$$\sum_{k=0}^{L-N} P_k = 1, \quad \sum_{k=1}^{L-N} kP_k = \frac{1}{\rho} - 1 \quad (7)$$

we should truncate Eqs. (6) and obtain other $L-N-1$ equations. In the present paper, we will choose the $L-N-1$ equations with m from 0 to $L-N-2$. Thus, we will have $L-N+1$ independent equations, from which one can readily work out the values of $P_0, P_1, \cdots P_{L-N}$, then the average speed of the traffic in the steady state is:

$$\begin{aligned} \langle v(t \to \infty) \rangle &= \sum_{i=1}^{M} P_i[(i-1)f + i(1-f)] + \sum_{i=M+1}^{L-N} P_i[(M-1)f + M(1-f)] \\ &= \sum_{i=1}^{M} iP_i + \sum_{i=M+1}^{L-N} MP_i - f(1-P_0) \end{aligned} \quad (8)$$

Based on the above discussions, one can easily obtain the traffic flux of the steady state:

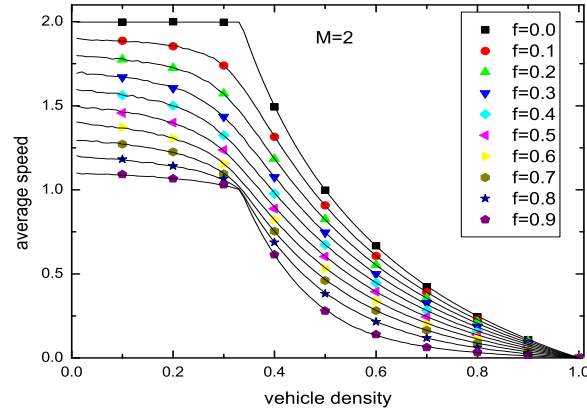$$J(t \to \infty) = \rho \langle v(t \to \infty) \rangle. \quad (9)$$

FIG. 1: The fundamental diagram of the average speed with the maximum speed $M = 2$ and for different stochastic delay probabilities f. The solid curves are numerical simulations while the points with different symbols represent theoretical results. The curves from the top down along the vehicle velocity axis correspond to different values of f ranging from 0 to 0.9, in steps of 0.1.
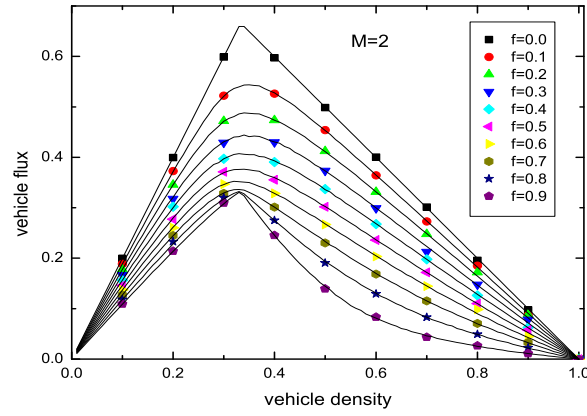

FIG. 2: The fundamental diagram of the vehicle flux with the maximum speed $M = 2$ and for different stochastic delay probabilities f. The solid curves are numerical simulations while the points with different symbols represent theoretical results. The curves from the top down along the traffic flux axis correspond to different values of f ranging from 0 to 0.9, in steps of 0.1.

III. NUMERICAL SIMULATIONS IN COMPARISON WITH THEORETICAL RESULTS

In order to compare with the analytical results, we performed numerical simulations on a 1D CA with length $L = 1000$ and the maximum car velocity $M = 2$. The number of vehicles N is adjusted so as to give the desired vehicle density ρ. The first 5000 time steps are excluded from the averaging procedure so as to remove the transient behavior. The average values are taken over the next 1000 time steps. Figure 1 and 2 show comparisons between results obtained from numerical simulations and car-oriented mean-field theory over the entire range of the vehicle density ρ. The curves are the simulation results, while the symbols represent the theoretical results. Theoretical results are in excellent agreement with numerical simulations.

IV. DISCUSSION

In summary, we propose and study a one-dimensional CA model of high-speed vehicles with the FI acceleration rule and the NS stochastic delay mechanism. The analysis of the dynamical evolution of our model give us a clearly physical picture of how the acceleration and stochastic delay rules affect the evolution and the corresponding asymptotic steady state. Although cellular automata are designed for efficient computer simulation studies, an analytical description is possible, though difficult all too often. The method presented here can also be applied to other CA models. Our study may shed some new light on developing analytical approaches to other one-dimension traffic flow models, especially the NS model, for which until now, no exact analytical approach has been established. The investigation of the NS model has lead to a better understanding of its advantages and limitations, so that it is easier to choose the approach most suitable for a given problem. The question, how the stationary state is approached, is still an important open issue, which is currently under investigation and promises to yield new insights into the physics of the NS model.

Acknowledgments

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